

Density Perturbation Growth in Teleparallel Cosmology

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ABSTRACT: We study the cosmological perturbations in teleparallel dark energy models in which there is a dynamical scalar field with a non-minimal coupling to gravity. We find that the propagating degrees of freedom are the same as in quintessence cosmology despite that variables of the perturbed vierbein field are greater than those in metric theories. The resulting growth evolution shows that gravitational interactions are enhanced during the unique tracker evolution of teleparallel dark energy models.

KEYWORDS: Modified gravity, dark energy, cosmological perturbation

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1 Introduction

The dark energy mystery has received much attention in the context beyond simply embedding the cosmological constant in general relativity (GR). To avoid a gigantic fine-tuning issue when the cosmological constant is attributed to the vacuum energy density, a canonical scalar field is introduced with the dynamical behavior known as quintessence [1–6]. Further modification of quintessence has been considered by allowing a non-minimal coupling between the scalar field and gravity [7–15] as well as either using the “wrong” sign phantom field [16–21], or combining a non-canonical extension such as the scalar-tensor theories [22, 23].

Recently, the “teleparallel” description of GR [24, 25] reveals an interesting aspect of gravity to explain the dark energy origin. The late-time cosmic acceleration has been seen from the generalized action of teleparallel equivalence of GR (TEGR) [26, 27], named as $f(T)$ gravity where T denotes the gravity Lagrangian of TEGR. Although $f(T)$ gravity shows a clear reference to $f(R)$ gravity, it provides no higher than second-order field equations, and no trivial Einstein frame available through conformal transformations [28]. On the other hand, teleparallel gravity incorporating with a dynamical scalar field describes a different scenario to drive the accelerated expansion of the Universe. This class of teleparallel dark energy models [29–31] lies in the framework of TEGR where the simplest quintessence model [1] is recovered if the scalar field is minimally coupled to gravity. Given that the torsion scalar T differs from the Lagrangian density of GR by a total derivative, the presence of a non-minimal coupling between the scalar field and gravity leads to interesting cosmological behaviors [1, 7, 16], while teleparallel cosmology shows compelling results with current observations [32].

However, it is noteworthy that modified teleparallel gravity theories, including both $f(T)$ gravity and teleparallel dark energy models, do not respect to the Lorentz invariance in the local tangent frame. The Lorentz violation in the teleparallel formalism beyond TEGR inevitably introduces extra degrees of freedom, which are unfamiliar in the framework of GR. This fact is manifestly provided by the field equations, which address the dynamics of all 16 components of the vierbein field [33, 34]. Under the homogeneous and isotropic principles of the background cosmology, those extra degrees of freedom merely involve at perturbation level, but have significant effects on the structure formulation in the Universe [35].

In the present work, we study the cosmological perturbations in teleparallel dark energy models via the perturbed vierbein field, which contains some variables with no reference in metric perturbations [36]. Nevertheless, in the scalar perturbations, we find that only one new degree of freedom involves at linear level, and we obtain one implicit constraint equation to the scalar mode variables from the asymmetric part of the field equations. As a result, the new scalar mode is resolved with a mere algebraic relation to other scalar modes, and thus the propagating degrees of freedom in teleparallel cosmology do not increase.

We proceed the density perturbations in sub-horizon scales to study the growth behaviors of some specific teleparallel dark energy models. For the simplest tracker solution corresponding to the non-minimal gravity-field coupling [37] (hereafter we denote as “NGF tracker”), we compare the growth rate from the effective gravitational constant to the GR case under an identical evolution background. We find that the NGF tracker always dominates, even in the presence of potentials of the scalar field, if the coupling between gravity and the scalar field is large to the order of unity. In the broader case with a smaller non-minimal coupling, the cosmic acceleration can be driven by the potential or the NGF tracker, depending on the initial composite differences of dark energy. The resulting density perturbation growth indicates that gravitational interactions are stronger if the evolution follows the NGF tracker rather than the usual potential-driven cosmic acceleration case.

This paper is organized as follows. In Sec. II, we review teleparallel dark energy theories. In Sec. III, we demonstrate the perturbation equations and apply the results to study the matter growth in Sec. IV. Finally, conclusions are given in Sec. V.

2 Teleparallel Dark Energy

The teleparallel dark energy model is given by [29]:

$$S = \int d^4x e \left[\frac{T}{2\kappa^2} + \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + \xi T \phi^2 \right) - V(\phi) + \mathcal{L}_m \right], \quad (2.1)$$

where $e \equiv \det(e_\mu^A) = \sqrt{-g}$ with $\kappa^2 = 8\pi G$, ξ is the non-minimal coupling parameter, and

$$T = \frac{1}{4} T_\rho^{\mu\nu} T_{\mu\nu}^\rho - \frac{1}{2} T^{\mu\nu}{}_\rho T_{\mu\nu}^\rho - T_\rho^\rho T_\nu^{\nu\mu} \quad (2.2)$$

is the torsion scalar of TEGR specially constructed with the quadratics of the torsion tensor [26]:

$$T_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda = e_A^\lambda (\partial_\mu e_\nu^A - \partial_\nu e_\mu^A). \quad (2.3)$$

It is noticeable that the connection $\Gamma_{\mu\nu}^\lambda = e_A^\lambda \partial_\nu e_\mu^A$ used in TEGR not only describes a curvature-less geometry but also guarantees the independent parallel transformation among the four unit vectors $\mathbf{e}_A(x^\mu)$, which form an orthonormal basis of the tangent space: $\mathbf{e}_A \cdot \mathbf{e}_B = \eta_{AB}$, where $\eta_{AB} = \text{diag}(1, -1, -1, -1)$ [25]. These vectors, \mathbf{e}_A , are commonly addressed by the components e_A^μ in the coordinate basis, *i.e.* $\mathbf{e}_A = e_A^\mu \partial_\mu$ ¹, while the metric tensor is given by the dual vierbein as $g_{\mu\nu} = \eta_{AB} e_\mu^A e_\nu^B$.

The field equations are obtained from the variation with respect to the vierbein, e_μ^A , which is the dynamical field of teleparallel gravity, given by

$$\begin{aligned} \left(\frac{1}{\kappa^2} + \xi \phi^2 \right) G_A^\nu - e_A^\nu \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + e_A^\mu \partial^\nu \phi \partial_\mu \phi \\ + 4\xi S_A^{\lambda\nu} \phi (\partial_\lambda \phi) = \Theta_A^\nu, \end{aligned} \quad (2.4)$$

where $\Theta_A^\nu \equiv e^{-1} \delta \mathcal{L}_m / \delta e_\nu^A$ denotes the energy-momentum tensor of the matter source, and

$$G_A^\nu = 2e^{-1} \partial_\mu (e e_A^\rho S_\rho^{\mu\nu}) - 2e_A^\lambda T^\rho_{\mu\lambda} S_\rho^{\nu\mu} - \frac{1}{2} e_A^\nu T$$

is nothing but the equivalent geometrical structure of the Einstein tensor with $G_{\mu\nu} = e_\mu^A G_{A\nu}$, where

$$S_\rho^{\mu\nu} = \frac{1}{2} \left(K^{\mu\nu}_\rho + \delta_\rho^\mu T^{\alpha\nu}_\alpha - \delta_\rho^\nu T^{\alpha\mu}_\alpha \right)$$

with $K^{\mu\nu}_\rho = -\frac{1}{2} (T^{\mu\nu}_\rho - T^{\nu\mu}_\rho - T_\rho^{\mu\nu})$. We can see that in the minimal limit ($\xi = 0$), the field equation (2.4) is reduced to the simplest quintessence model.

Torsion in (2.3) is defined by a Lorentz violated formalism for the tangent frame [26], which provides a technically simpler approach to acquire results of the cosmological interest [38]. The lack of Lorentz symmetry in teleparallelism results in extra degrees of freedom [33] as seen from (2.4), which in fact exhibits the equation of motion of all the 16 degrees of freedom of the vierbein field. To be more precise, given that the energy-momentum tensor $\Theta_{\mu\nu}$ and the Einstein tensor $G_{\mu\nu}$ are both symmetric, the anti-symmetrization between indices A and ν in (2.4) leads to a non-trivial constraint

$$4\xi \phi \left(g^{\mu\alpha} S_\mu^{\lambda\beta} - g^{\nu\beta} S_\nu^{\lambda\alpha} \right) \partial_\lambda \phi = 0, \quad (2.5)$$

which forms a system of 6 equations in addition to the usual 10 equations from the symmetric part of the field equations. Incidentally, this constraint vanishes if $\xi = 0$ so that TEGR shares the same number of dynamical degrees of freedom as GR.

Nevertheless, we will see in the later discussion that only one extra scalar mode beyond the metric perturbation variables will be involved in the linear perturbations of teleparallel dark energy theories. However, this new degree of freedom is not dynamically independent due to the constraint in (2.5).

¹We use the notations as follows: Greek indices μ, ν, \dots and capital Latin indices A, B, \dots run over all coordinate and tangent space-time 0, 1, 2, 3, while lower case Latin indices (from the middle of the alphabet) i, j, \dots and lower case Latin indices (from the beginning of the alphabet) a, b, \dots run over spatial and tangent space coordinates 1, 2, 3, respectively.

The flat Friedmann-Robertson-Walker (FRW) background metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$$

is conventionally resolved by the background vierbein choice

$$e_\mu^A = \text{diag}(1, a, a, a), \quad (2.6)$$

where $a(t)$ is the scale factor. The Friedmann equations are given by

$$H^2 = \frac{\kappa^2}{3}(\rho_\phi + \rho_m), \quad (2.7)$$

$$\dot{H} = -\frac{\kappa^2}{2}(\rho_\phi + p_\phi + \rho_m + p_m), \quad (2.8)$$

where

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) - 3\xi H^2 \phi^2, \quad (2.9)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) + 4\xi H \phi \dot{\phi} + \xi (3H^2 + 2\dot{H}) \phi^2, \quad (2.10)$$

with $H = \dot{a}/a$ the Hubble parameter. The variation of the action with respect to the scalar field yields

$$\square\phi - \xi T\phi + V_\phi = 0, \quad (2.11)$$

where $V_\phi \equiv dV/d\phi$. Consequently, the background equation motion of ϕ is given by

$$\ddot{\phi} + 3H\dot{\phi} + 6\xi H^2 \phi + V_\phi = 0. \quad (2.12)$$

The continuity equation $\dot{\rho} + 3H(1 + w)\rho = 0$ holds for both ρ_m and ρ_ϕ , where $w \equiv p/\rho$ is the equation of state parameter. It is interesting to note that, due to the lack of trivial Einstein frame available through conformal transformations, the effective equation of state of dark energy $w_\phi = p_\phi/\rho_\phi$ in teleparallel cosmology can exhibit [29] quintessence-like, phantom-like or phantom-divide crossing evolutions, respectively.

3 Perturbation Equations

The general vierbein field has 16 components, which are dynamical in teleparallel gravity theories. In particular, 6 of the 16 degrees of freedom, arising as a result of the non-invariance of the local Lorentz symmetry, are unfamiliar in the metric perturbation scenario, and shall not appear in the field equation of the minimal TEGR limit. The perturbed vierbein field is given by [36]

$$\begin{aligned} e_\mu^0 &= \delta_\mu^0(1 + \psi) + a\delta_\mu^i \partial_i(F + \alpha) + a\delta_\mu^i (G_i + \alpha_i), \\ e_\mu^a &= a\delta_\mu^a(1 - \varphi) + a\delta_\mu^i (\partial_i \partial^a B + \partial^a C_i + h^a_i) \\ &\quad + a\delta_\mu^i B^a_i + \delta_\mu^0 (\partial^a \alpha + \alpha^a), \end{aligned} \quad (3.1)$$

where B^a_i is an antisymmetric spatial tensor with $B_{ij} + B_{ji} = 0$ and $B_{ij} \equiv \delta^a_i B_{aj}$, while C_i , G_i , and α_i are transverse vectors with $\partial^i C_i = \partial^i G_i = \partial^i \alpha_i = 0$. It is easy to obtain the corresponding metric from (3.1) as:

$$\begin{aligned} g_{00} &= 1 + 2\psi \\ g_{i0} &= a(\partial_i F + G_i) \\ g_{ij} &= -a^2[(1 - 2\varphi)\delta_{ij} + h_{ij} + \partial_i \partial_j B + \partial_j C_i + \partial_i C_j], \end{aligned} \quad (3.2)$$

which show the familiar metric perturbations in GR. Hence, we observe that the unfamiliar modes in the perturbed vierbein (3.1) are the scalar α , the vector α^i and the tensor B^a_i , which comprise $1 + 2 + 3 = 6$ degrees of freedom. Namely, the 16 components of the vierbein field are illustrated by the 5 scalar modes: ψ , ϕ , B , F and α , 3 vector modes: C_i , G_i and α_i , an antisymmetric tensor B^a_i , and a transverse traceless tensor mode h_{ij} , respectively.

In the following study, we focus on the cosmological implication of the scalar perturbations. To simplify the calculations, it is convenient to eliminate the variables F , B and C_i through the general coordinate transformations $x^\mu \rightarrow x^\mu + \epsilon^\mu(x)$. The vierbein perturbation is provided by [36], given by

$$\begin{aligned} e^0_\mu &= \delta^0_\mu(1 + \psi) + a\delta^i_\mu \partial_i \alpha \\ e^a_\mu &= a\delta^a_\mu(1 - \varphi) + a\delta^i_\mu B^a_i + \delta^0_\mu \partial^a \alpha, \end{aligned} \quad (3.3)$$

leading to the perturbed metric in terms of the longitudinal gauge form:

$$ds^2 = (1 + 2\psi)dt^2 - a^2(1 - 2\varphi)\delta_{ij}dx^i dx^j.$$

Subsequently, we obtain the torsion tensor and scalar to be

$$\begin{aligned} T^0_{0i} &= -\partial_i \psi + a\partial_0 \partial_i \alpha, \\ T^i_{0j} &= (H - \dot{\varphi})\delta^i_j + \partial_0 B^i_j - a^{-1}\partial_j \partial^i \alpha, \\ T^i_{jk} &= \partial_k(\delta^i_j \varphi - B^i_j) - \partial_j(\delta^i_k \varphi - B^i_k), \end{aligned} \quad (3.4)$$

and

$$T = -6H^2 + 12H(\dot{\varphi} + H\psi) + 4a^{-2}\partial^2 \alpha_m, \quad (3.5)$$

respectively, where $\alpha_m \equiv aH\alpha$.

On the other hand, we can also decompose the scalar field to homogeneous and perturbed parts: $\phi \rightarrow \phi(t) + \delta\phi(t, x^i)$. The scalar perturbations of the matter source Θ^μ_ν are denoted by the perfect fluid perturbations: $\Theta^0_0 = -(\rho_m + \delta\rho)$, $\Theta^0_i = -(\rho_m + p_m)\partial_i \delta u$ and $\Theta^i_j = (p_m + \delta p)\delta^i_j$, where δu is the scalar vector potential of the fluid.

After substituting these expressions to the field equation (2.4), we derive the perturbation equations given by the energy density Θ^0_0 :

$$(1 + \kappa^2 \xi \phi^2) \left[6H(\dot{\varphi} + H\psi) - 2\frac{k^2}{a^2} \varphi \right] + \kappa^2 \left(V_\phi \delta\phi - \dot{\phi}^2 \psi - 6\xi H^2 \phi \delta\phi \right) = -\kappa^2 \delta\rho, \quad (3.6)$$

the energy flux Θ_0^i :

$$\begin{aligned} & 2(1 + \kappa^2 \xi \phi^2) \partial^i (\dot{\varphi} + H\psi) - \kappa^2 \dot{\phi} \partial^i \delta\phi + 4\kappa^2 \xi \phi \dot{\phi} \left(\partial^i \varphi + \frac{1}{2} \partial_j B^{ji} \right) \\ & = -\kappa^2 (\rho_m + p_m) \partial^i \delta u, \end{aligned} \quad (3.7)$$

and the momentum density Θ_i^0 :

$$-2(1 + \kappa^2 \xi \phi^2) \partial_i (\dot{\varphi} + H\psi) + \kappa^2 \dot{\phi} \partial_i \delta\phi + 4\kappa^2 \xi H \phi \partial_i \delta\phi = \kappa^2 (\rho_m + p_m) \partial_i \delta u, \quad (3.8)$$

respectively, while the momentum flux Θ_i^j is divided by its diagonal components ($i = j$):

$$\begin{aligned} & (1 + \kappa^2 \xi \phi^2) \left[12H(\dot{\varphi} + H\psi) + 2(\ddot{\varphi} + H\dot{\psi} + 2\dot{H}\psi) \right] \\ & - \kappa^2 \left[\dot{\phi} \delta\dot{\phi} - V_\phi \delta\phi - \dot{\phi}^2 \right] + 4\kappa^2 \xi \phi \dot{\phi} (\dot{\varphi} + 2H\psi) - 4\kappa^2 \xi H \dot{\phi} \delta\phi = \kappa^2 \delta p, \end{aligned} \quad (3.9)$$

as well as its anisotropic parts ($i \neq j$):

$$\psi = \varphi - \frac{2\kappa^2 \xi \phi \dot{\phi}}{H(1 + \kappa^2 \xi \phi^2)} \alpha_m. \quad (3.10)$$

Similarly, the perturbed equation of motion for the scalar field (2.11) is given by

$$\begin{aligned} & \delta\ddot{\phi} + 3H\delta\dot{\phi} + \left(\frac{k^2}{a^2} + V_{\phi\phi} \right) \delta\phi + 6\xi H^2 \delta\phi - 2(\ddot{\phi} + 3H\dot{\phi})\psi - \dot{\phi}(3\dot{\varphi} - \dot{\psi}) \\ & - 12\xi \phi (\dot{\varphi} + H\psi) - 4\xi \phi \frac{k^2}{a^2} \alpha_m = 0. \end{aligned} \quad (3.11)$$

To complete the perturbation equations of teleparallel dark energy models, we shall also include equations from the constraint (2.5). In linear perturbations, this constraint yields

$$\dot{\phi} \partial^i \varphi + \frac{1}{2} \dot{\phi} \partial_j B^{ji} + H \partial^i \delta\phi = 0, \quad (3.12)$$

which is necessary for the consistent relation of the matter source $\Theta_{0i} = \Theta_{i0}$, as also indicated from the comparison between (3.7) and (3.8). Given that the antisymmetric tensor B_{ij} satisfies $\partial_i \partial_j B^{ij} = 0$, we find that the gradient of (3.12) leads to

$$\varphi = -\frac{H}{\dot{\phi}} \delta\phi. \quad (3.13)$$

Note that this equation holds even for the absence of the matter source $\Theta_{\mu\nu}$ with the non-minimal coupling $\xi \neq 0$ of the theory (2.1).

4 Matter Growth

Although teleparallel gravity gives a gravitational concept different from GR, the perturbations of the matter sector shall be treated in the usual manner regardless to the geometrical background. In particular, the equation motion of a particle in the presence of gravity in

TEGR is mathematically equivalent to the geodesic equation in GR [26], while the conservation of energy-momentum $\nabla^\mu \Theta_{\mu\nu} = 0$ is the covariant derivative with respect to the GR connection [33, 39]. Thus, in teleparallel cosmology, the gauge-invariant matter density perturbation of the dust-like source, $\delta_m \equiv \delta\rho_m/\rho_m + 3H\delta u$, also satisfies the perturbation equation

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}}\rho_m\delta_m = 0, \quad (4.1)$$

where the effective gravitational coupling constant G_{eff} coincides to the Newton's constant in the minimal case of $\xi = 0$. In the discussion, we consider the Fourier space by simply replacing the operator ∂^2 with the wave number k^2 . In the sub-horizon regime $k \gg aH$, the perturbed equation of $\nabla^\mu \Theta_{\mu\nu} = 0$ can lead to a useful relation $4\pi G_{\text{eff}}\rho_m\delta_m \simeq k^2\psi/a^2$ under the longitudinal gauge. The effective gravitational coupling G_{eff} can be obtained once the relation between ψ and δ_m is found.

To proceed a further simplification, we adopt the quasi-static approximation

$$|\dot{X}| \lesssim |HX|, \quad \text{for } X = \phi, \psi, \varphi,$$

to the perturbation equations (3.6)-(3.11). Together with the sub-horizon condition $k \gg aH$, we find that (3.11) becomes

$$\left(\frac{k^2}{a^2} + V_{\phi\phi}\right)\delta\phi - 4\xi\phi\frac{k^2}{a^2}\alpha_m \simeq 0, \quad (4.2)$$

while (3.6) gives

$$-2(1 + \kappa^2\xi\phi^2)\frac{k^2}{a^2}\varphi + \kappa^2V_\phi\delta\phi \simeq -\kappa^2\delta\rho, \quad (4.3)$$

where $|\xi|$ is $O(10^{-1})$ according to the observational constraints [32]. We shall further neglect the contributions of V_ϕ and $V_{\phi\phi}$ as for the general scalar-tensor theory of dark energy [40], and it can be checked that any potential for teleparallel dark energy indeed satisfies $k^2 \gg a^2\kappa V_\phi$ and $k^2 \gg a^2V_{\phi\phi}$ in the sub-horizon regime [32].

From (3.13) and (4.2), we obtain that $\delta\phi = -\dot{\phi}\varphi/H \simeq 4\xi\phi\alpha_m$. The relation (3.10) results in

$$\psi = \left(1 + \frac{\epsilon}{1 + \kappa^2\xi\phi^2}\right)\varphi, \quad (4.4)$$

where $\epsilon \equiv \kappa^2\dot{\phi}^2/2H^2$. From (4.3) and (4.4), we finally arrive at

$$\frac{k^2}{a^2}\psi = \frac{1}{1 + \kappa^2\xi\phi^2} \left(1 + \frac{\epsilon}{1 + \kappa^2\xi\phi^2}\right) \frac{\kappa^2}{2}\delta\rho, \quad (4.5)$$

which implies

$$G_{\text{eff}} = \left(1 + \frac{\epsilon}{1 + \kappa^2\xi\phi^2}\right) \frac{G}{1 + \kappa^2\xi\phi^2}. \quad (4.6)$$

Note that ϵ is proportional to the kinetic energy density of the scalar field, which is subdominated to the energy density of dark energy Ω_ϕ at the present in teleparallel dark energy models. In the minimal quintessence limit $\xi = 0$, the cosmic acceleration in fact requires $\epsilon \ll \Omega_\phi$ and the effective gravitational coupling reduces to the expected result, *i.e.* $G_{\text{eff}} = (1 + \epsilon)G \simeq G$.

4.1 Purely gravity coupling $V = 0$

In the following, we exam the matter growth in teleparallel dark energy models with specific forms of potentials. The simplest tracker behavior has been found in the potential-less case ($V = 0$) where one can achieve the late-time cosmic acceleration for $\xi < 0$, while the observational constraints from type Ia supernova (SNIa), baryon acoustic oscillation (BAO) and cosmic microwave background (CMB) data reveal a theoretically viable region of $-1 < \xi < -0.125$ with the best-fit value at $\xi = -0.35$ [37]. To analyze the evolution in this case, we re-parametrize two new variables: $x = \kappa\dot{\phi}/\sqrt{6}H$ and $y = \sqrt{-\xi}\kappa\phi$ so that the Friedmann equation (2.7) is rewritten as

$$1 = \Omega_m + x^2 + y^2. \quad (4.7)$$

The evolutions of x and y with respect to $N = \ln a$ are given by

$$\begin{aligned} \frac{dx}{dN} &= -\left(3 + \frac{\dot{H}}{H^2}\right)x + \sqrt{-6\xi}y, \\ \frac{dy}{dN} &= \sqrt{-6\xi}x. \end{aligned} \quad (4.8)$$

Note that (2.8) also provides a useful relation:

$$(1 - y^2)\frac{\dot{H}}{H^2} = -3x^2 + 2\sqrt{-6\xi}xy - \frac{3}{2}\Omega_m, \quad (4.9)$$

to solve the evolutions numerically from some given initial conditions.

By using $\mathcal{G} \equiv d\ln(\delta/a)/d\ln a$ to re-parametrize the matter perturbations, the second derivative equation (4.1) is reduced to

$$\frac{d\mathcal{G}}{dN} + \left(2 + \frac{\dot{H}}{H^2}\right)(\mathcal{G} + 1) + (\mathcal{G} + 1)^2 = \frac{3}{2}Q\Omega_m, \quad (4.10)$$

where $Q \equiv G_{\text{eff}}/G$. Subsequently, from (4.6) we find that

$$Q = \frac{1}{1 - y^2} \left(1 + \frac{\sqrt{6}x^2}{2(1 - y^2)}\right). \quad (4.11)$$

We remark that (4.10) is applicable to the generic modified gravity theories if the effective coupling G_{eff} is given. In the GR case, the contribution of the \mathcal{G}^2 term in (4.10) is negligible throughout the growth history from matter domination [41]. Here, we adopt the same approximation given that the present value of \mathcal{G} is not smaller than -0.5 for all of the considered cases in this work. As a result, we can have the evolution of \mathcal{G} solved simultaneously with the other parameters x and y from given initial conditions. In Fig. 1, we show the evolution of the growth rate δ as a function of the redshift z in the purely gravity coupling teleparallel dark energy model ($V = 0$), where the dotted, red-solid and dot-dashed lines correspond to the observational lower bound, best-fit and upper bound values of $\xi = -1$, -0.35 and -0.125 respectively, while the green-solid line represents

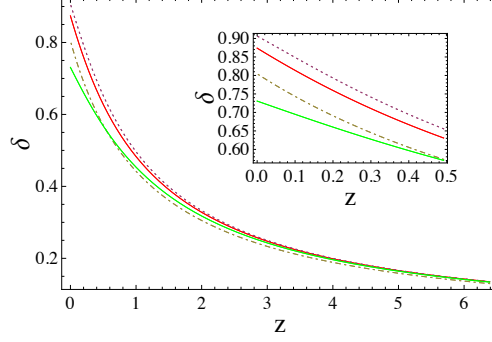


Figure 1. Growth rate δ as a function of the redshift z in the purely gravity coupling teleparallel dark energy model ($V = 0$), where the dotted, red-solid and dot-dashed lines correspond to the lower, best-fit and upper values of $\xi = -1, -0.35$ and -0.125 , respectively, while the green-solid line represents $\xi = -0.35$ with a fixed value of $Q = 1$.

$\xi = -0.35$ with a fixed value of $Q = 1$ so that gravitational interaction mimics that of GR. The initial values provided are $\{\Omega_m = 0.999, y = 1.0 \times 10^{-6}, \mathcal{G} = 0\}$ for $z \gg 1$. The results in Fig. 1 indicate that the gravitational interaction is in fact enhanced ($Q > 1$) during the evolution, which ensures a more reasonable approximation by dropping the \mathcal{G}^2 term in (4.10).

4.2 Exponential potential $V = V_0 e^{-2\kappa\lambda\phi}$

We can extend the scenario to teleparallel dark energy models with non-vanished potentials. As an illustration, we concentrate on the exponential type, $V = V_0 e^{-2\kappa\lambda\phi}$, which has a constant potential limit $V = V_0$ if $\lambda = 0$. We shall introduce a new parameter, $v \equiv \kappa\sqrt{V}/\sqrt{3}H$, to address the evolution of the potential. The Friedmann equation (2.7) is rewritten as

$$1 = \Omega_m + x^2 + y^2 + v^2, \quad (4.12)$$

where x and y share the same definitions as the previous case. The evolution equations of these components are given by

$$\begin{aligned} \frac{dx}{dN} &= -\left(3 + \frac{\dot{H}}{H^2}\right)x + \sqrt{-6\xi}y + \sqrt{6}\lambda v^2, \\ \frac{dy}{dN} &= \sqrt{-6\xi}x, \\ \frac{dv}{dN} &= -\left(\sqrt{6}\lambda x + \frac{\dot{H}}{H^2}\right)v. \end{aligned} \quad (4.13)$$

We consider the regime of $\xi < 0$ for the exponential potential since it is favored from SNIa, BAO and CMB observations [32] and solve x , y and v together with the density perturbation $\mathcal{G} = d\ln(\delta/a)/d\ln a$ in (4.10). Note that (4.9) is unchanged.

In the case with the non-minimal coupling of $\xi \lesssim -0.01$, the result (refer to the green-solid line in Fig. 2) is similar to the quintessence model where the potential dominates

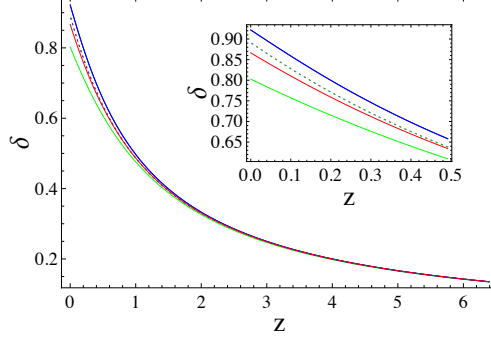


Figure 2. Growth rate δ as a function of z in the teleparallel dark energy model with $V = V_0 e^{-2\kappa\lambda\phi}$, where the red-solid, blue-solid and green-solid lines correspond to $\xi = -0.4, -1$ and -0.125 with the initial values of $\{\Omega_m = 0.999, v = 1.0 \times 10^{-6}\}$, and the dotted line depicts $\xi = -0.4$ with the initial values $\{\Omega_m = 0.990, v = 6.0 \times 10^{-3}\}$, while we have fixed $\lambda = 0.5$ and the initial values for $z \gg 1$ to be $\{y = 1.0 \times 10^{-6}, \mathcal{G} = 0\}$.

during the evolution. However, if ξ is around the best-fit value of ~ -0.4 [32], the purely gravity coupling tracker behavior appears when the given initial value of v is sufficiently small, as illustrated by the dotted line in Fig. 2 with the initial value $v \sim 10^{-6}$. The comparison to the red-solid line shows that the gravitational interaction is weaker if the cosmic acceleration is driven by the potential instead of the NGF tracker effect. On the other hand, if the non-minimal coupling ξ is ~ -1 as described by the blue-solid line in Fig. 2, then only NGF tracker behavior is found regardless to the initial values of the potential. Clearly, the parameter y will finally dominate the dark energy density Ω_ϕ at the present.

5 Conclusions

In this work, we have applied the vierbein perturbation scenario to study the cosmological perturbations in teleparallel dark energy models with the non-minimal coupling between the scalar field and gravity. Unfamiliar with the metric perturbations, a new scalar mode, α_m , appears in the linear equations and mainly contributes to the anisotropic between the gravitational potentials ψ and φ . This new scalar degree of freedom is fully suppressed in the sub-horizon scales of $f(T)$ gravity [36], but has significant contributions in the large scale structures [35]. Nevertheless, the dynamical degrees of freedom of the theory do not increased by α_m given that the constraint in (2.5) provides an additional algebraic relation to the scalar mode variables.

To study the matter growth in teleparallel cosmology, we have used the quasi-static approximation in the sub-horizon scales. The effective gravitational coupling becomes scale independent in the sub-horizon regime similar to the generic scalar-tensor theories. In the case where the cosmic acceleration is driven by the non-minimal gravity-field coupling term, i.e. the $\xi H^2 \phi^2$ term in (2.9), we have found that the gravitational interaction is enhanced so that matter grows faster than that in GR. In the broader case where the scalar field

has a potential, the quintessence behavior recovers with a small coupling limit of ξ , while the non-minimal gravity-field tracker is followed when $\xi \sim -1$. In the case with ξ around the best-fit value (~ -0.4) from observations, we can also observe a faster growth rate if the NGF tracker is followed instead of the usual quintessence-like evolution. These results feature the matter perturbations in teleparallel dark energy models and worth for further investigations to look for the possible deviations from GR.

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